3.5 $A \subset \mathbb{R}$, A' 至多可数，则 A 至多可数

$A \setminus A'$ 无极限点，否则该点属于 A' ，矛盾。
在 $C2$ 空间（有可数基），孤立点集至多可数，故 $A \setminus A'$ 至多可数，进而 A 至多可数。

3.6 $[0,1]$ cannot be the disjoint union of countably many closed sets.

Lemma 1 : $F \subset \mathbb{R}$ is a closed set. C is a component of F. Then $\partial F \cap C \neq \phi$

Let $x_0 \in C$. Suppose that C is disjoint from ∂F. Then there is an open-closed set $A \subset Int(F) \subset F$ and $x_0 \in A$. Then by the definition of subspace topology, $\exists U \subset \mathbb{R}$, which is open, such that $A = U \cap F$. Since $A \cap Int(F) = \phi$, we have $A = U \cap Int(F)$. Thus A is open in $\mathbb{R}$. But A is also closed in F, and F closed in $\mathbb{R}$. So A is closed in $\mathbb{R}$, which implies $A = \mathbb{R}$. And it is impossible.

Lemma 2 : Closed interval $F = \bigsqcup F_i$, F_i's are non-empty. Then for all $i \neq j$, there is a closed interval $C \subset \mathbb{R}$ satisfying $C \cap F_i = \phi, C \cap F_j \neq \phi$, and $C \cap F_k, k = 1, 2, \cdots$ has at least two non-empty sets.

If F_i is empty then we take $C = \mathbb{R}$. Thus we can assume that F_i's are not empty. Take $j \neq i$. By the property of (T4), there are disjoint open set U, V such that $F_i \subset U$ and $F_j \subset V$. Let $x \in F_j$ and C the component of x in the subspace $\overline{V} \subset U^C$. C is a closed interval and $C \cap F_i = \phi, C \cap F_j \neq \phi$. By lemma 1, $C \cap \partial \overline{V} \neq \phi$. Because $F_j \subset V$, there is another k such that F_k intersects with $C \cap \partial \overline{V}$, thus it intersects with C.

Let a closed interval $F = \bigsqcup F_i$, and each F_i is closed. For each i, we can find a closed interval $C_i \subset C_{i-1}$ which doesn't intersect with $\bigsqcup_{k=1}^{i} (F_i \cap C_{i-1})$. So we get a decreasing closed set sequence $C_1 \supset C_2 \supset \cdots$, thus $\cap_{k=1}^{\infty} C_k \neq \phi$. While $\cap_{k=1}^{\infty} C_k \cap F = \phi$ by the construction, which is a contradiction.

Remarks :

1. 注意区分闭集和闭区间
2. 开集可以写成可数个开区间的并，但闭区间不行

3.9 $\mathbb{Q}$ is not a G_δ set, i.e. $\mathbb{Q}$ cannot be the intersection of countably many open sets.

Suppose $\mathbb{Q} = \cap_{i=1}^{\infty} U_i = \{q_1, q_2, \cdots\}$, where U_i's are open sets. Take a closed interval $[a_1 - \epsilon_1, a_1 + \epsilon_1] \subset U_1 \setminus \{q_1\}$. Similarly we can find a closed interval $[a_k - \epsilon_k, a_k + \epsilon_k] \subset (U_k \setminus \{q_k\}) \cap (a_{k-1} - \epsilon_{k-1}, a_{k-1} + \epsilon_{k-1})$. By the compactness, $V = \cap_{k=1}^{\infty} [a_k - \epsilon_k, a_k + \epsilon_k] \neq \phi$. Because $V \subset U_i$ for each i, $U \subset \mathbb{Q}$. But $q_k \notin V$ by the construction.

Remarks:

1. Let $U_m = \cup_{n=1}^{\infty} (q_n - 1/2^{n+m}, q_n + 1/2^{n+m})$. Is $\cap_{m=1}^{\infty} U_m$ equal to $\mathbb{Q}$? What can we infer from this example?
2. There are "a lot of" points in U_n^C !

4.3 In (X, ρ), continuity is equivalent to sequence continuity.

"$\Rightarrow$" : $f : X \to Y$ is continuous. For any $x_n \to x_0$, and any neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0. Therefore $\exists N, \forall n > N, x_n \in f^{-1}(V) \Rightarrow f(x_n) \in V$. So $f(x_n) \to f(x_0)$.

"$\Leftarrow$" : It suffices to prove that for any $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. **By the property of metric space**, for each $a \in \overline{A}$, $\exists a_n \in A$ such that $a_n \to a$. Therefore $f(a_n) \to f(a)$, and $f(a) \in \overline{f(A)}$. So $f(\overline{A}) \subset \overline{f(A)}$.

Remarks :

1. Without "metric", $x \in \overline{A}$ maynot be the limit of a sequence of $x_n \in A$. For example, in $(\mathbb{R}, \mathcal{T}_{cocountable})$, let A be any uncountable set. Then $\overline{A} = \mathbb{R}$ since the intersection of A and every open set in $\mathbb{R}$ has uncountably many elements. But $x_n \to x$ if and only if $\exists N, \forall m, n > N, a_m = a_n$.

4.5.

(1°) $\{x \mid f(x) > c\} = f^{-1}((c,+\infty))$

$\{x \mid f(x) \geq c\} = f^{-1}([c,+\infty))$

$\{x \mid f(x) = c\} = f^{-1}(c)$

(2°) $(\Rightarrow)$ 显然

$(\Leftarrow)$ $f^{-1}((a,b)) = f^{-1}((a,+\infty) \cap (-\infty,b)) = f^{-1}((a,+\infty)) \cap f^{-1}((-\infty,b))$ 是开集，

而$\{(a,b) \mid a<b\}$是R上拓扑基. 从而f连续

对闭集，若 $\{x \mid f(x) \geq c\}$，$\{x \mid f(x) \leq c\}$ 均为闭集，则 $\{x \mid f(x) < c\}$，$\{x \mid f(x) > c\}$ 均为开集.

(3°)

$$f(x) = \begin{cases} 0 & x \neq 0 \\ -1 & x = 0 \end{cases}$$

4.6 (1°) $\forall E \subset Y$，$f^{-1}(E) \in \mathcal{T}_{离散}$.

(2°) $(\Rightarrow)$ 若 $G \cap f(X) \neq \phi$. 则 $f^{-1}(G) \neq \phi$. 而 $\mathcal{T}_{平凡} = \{\phi, X\}$.

从而只能有 $f^{-1}(G) = X$. $\Rightarrow f(X) \subset G$.

$(\Leftarrow)$ 显然

(3°) $(\Rightarrow)$ 若非常值映射，则 $f(X)$ 至少有两个元素 a, b.

取以 a 为中心的开球B，半径为 $r = \frac{1}{2} d(a,b)$.

则 $f(X) \not\subset B$. 从而 f 不连续

$(\Leftarrow)$ 显然.

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$(\Rightarrow)$ $\forall \varepsilon>0$

$V=\left(f(x_0)-\frac{\varepsilon}{2}, f(x_0)+\frac{\varepsilon}{2}\right)$

$f^{-1}(V)$ 是 x_0 的邻域 $\exists\, r>0$ s.t. $B(x_0, r)\subset f^{-1}(V)$

$\Rightarrow f(B(x_0,r))\subset V$

$\Rightarrow \omega_f(x_0,r)\leq \varepsilon$

$\Rightarrow \omega_f(x_0)\leq \varepsilon.$

$\Rightarrow \omega_f(x_0)=0.$

$(\Leftarrow)$ $\omega_f(x_0)=0.$

对于 $f(x_0)$ 的任何邻域 V. $\exists\, \varepsilon$ s.t. $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)\subset V$

$\exists\, \delta$ $\forall r\leq\delta$, 有 $\omega_f(x_0,r)<\varepsilon$

特别地有 $\omega_f(x_0,\delta)\leq\varepsilon$

$\Rightarrow f(B(x_0,\delta))\subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon)\subset V.$

$B(x_0,\delta)\subset f^{-1}(V).$

从而 f 在 x_0 处连续 $\square$